Dynamic programming in sparse graphs

Ignasi Sau

CNRS, LIRMM, Montpellier, France

Seminari de Teoria de Grafs, Combinatòria i Applicacions

Joint work with:

Juanjo Rué
Instituto de Ciencias Matemáticas, Madrid, Spain

Dimitrios M. Thilikos
Department of Mathematics, NKU of Athens, Greece

Outline

1. Motivation
2. Graphs on surfaces
   - Preliminaries
   - Main ideas of our approach
3. Extension to minor-free graphs
4. Conclusions
   - Further research
   - Some recent results

Next section is...

Some words on parameterized complexity

- Idea: given an NP-hard problem, fix one parameter of the input to see if the problem gets more "tractable".

  Example: the size of a VERTEX COVER.

- Given a (NP-hard) problem with input of size n and a parameter k, a fixed-parameter tractable (FPT) algorithm runs in

  $$f(k) \cdot n^{\mathcal{O}(1)}, \quad \text{for some function } f.$$

  Examples: k-VERTEX COVER, k-LONGEST PATH.

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FPT and single-exponential algorithms

- Courcelle's theorem (1988):

  Graph problems expressible in Monadic Second Order Logic can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$ in graphs with **tw** $\leq k$.

- **Problem**: $f(k)$ can be huge!!! (for instance, $f(k) = 2^{3^{4^{5^{6^{k}}}}}$)

- A single-exponential parameterized algorithm is a FPT algo s.t.

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> build a framework to obtain **single-exponential algorithms** for a class of NP-hard problems in **sparse graphs**.

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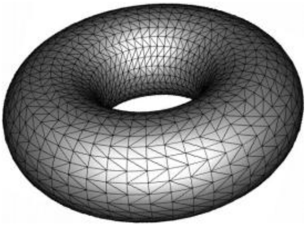
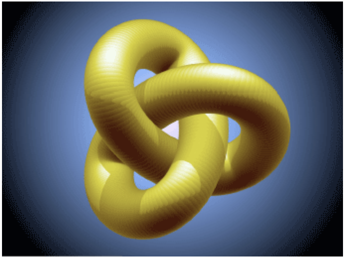
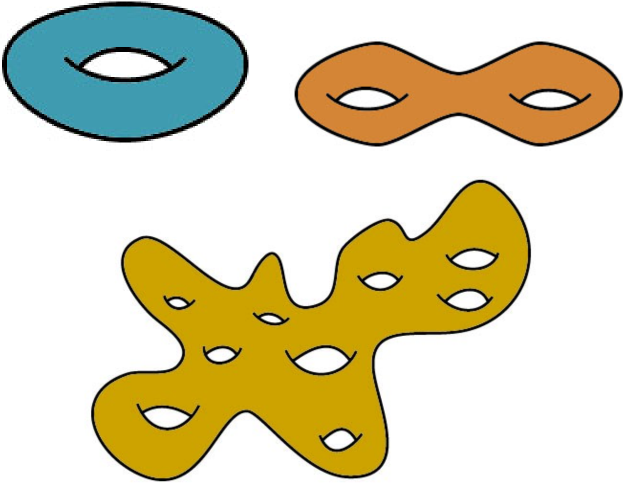
Surfaces

- **SURFACE** = TOPOLOGICAL SPACE, LOCALLY "FLAT"

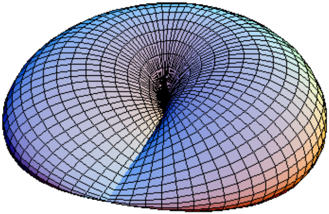


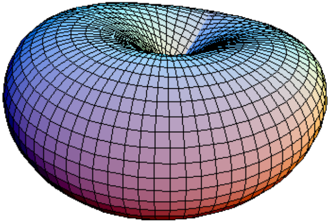


Handles



Cross-caps





Surface Classification Theorem

- **Surface Classification Theorem**:

  any compact, connected and without boundary surface can be obtained from the sphere $\mathbb{S}^2$ by adding handles and cross-caps.

- Orientable surfaces:

  obtained by adding $g \geq 0$ *handles* to the sphere $\mathbb{S}^2$, obtaining the g-torus $\mathbb{T}_g$ with Euler genus $\mathbf{eg}(\mathbb{T}_g) = 2g$.

- Non-orientable surfaces:

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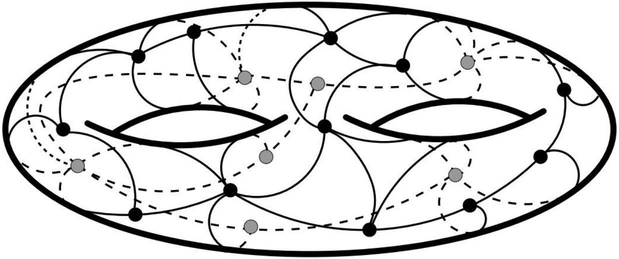

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Graphs on surfaces

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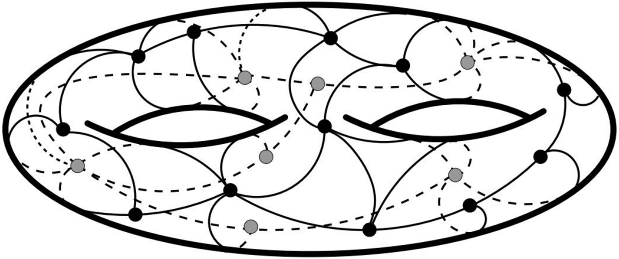

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Branch decompositions and branchwidth

- A branch decomposition of a graph $G = (V, E)$ is tuple (T, μ) where:
  - T is a tree where all the internal nodes have degree 3.
  - μ is a bijection between the leaves of T and $E(G)$.
- Each edge $e \in T$ partitions $E(G)$ into two sets A_e and B_e.
- For each $e \in E(T)$, we define $\mathbf{mid}(e) = V(A_e) \cap V(B_e)$.
- The width of a branch decomposition is $\max_{e \in E(T)} |\mathbf{mid}(e)|$.
- The branchwidth of a graph G (denoted $\mathbf{bw}(G)$) is the minimum width over all branch decompositions of G:

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- Applied in a bottom-up fashion on a rooted branch decomposition of the input graph G.
- For each graph problem, DP requires the suitable definition of tables encoding how potential (global) solutions are restricted to a middle set $\mathbf{mid}(e)$.
- The size of the tables reflects the dependence on $k = |\mathbf{mid}(e)|$ in the running time of the DP.
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1. A subset of vertices of **mid**(e) (not restricted by some global condition).
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   Examples: LONGEST PATH, CYCLE PACKING, HAMILTONIAN CYCLE.
   The # of pairings in a set of k elements is $k^{\Theta(k)} = 2^{\Theta(k \log k)}$...

   OK for planar graphs [Dorn, Penninkx, Bodlaender, Fomin. *ESA'05*];
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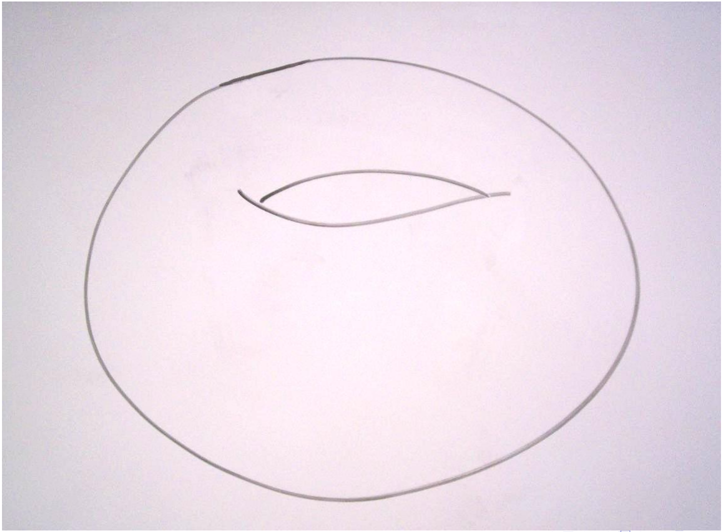

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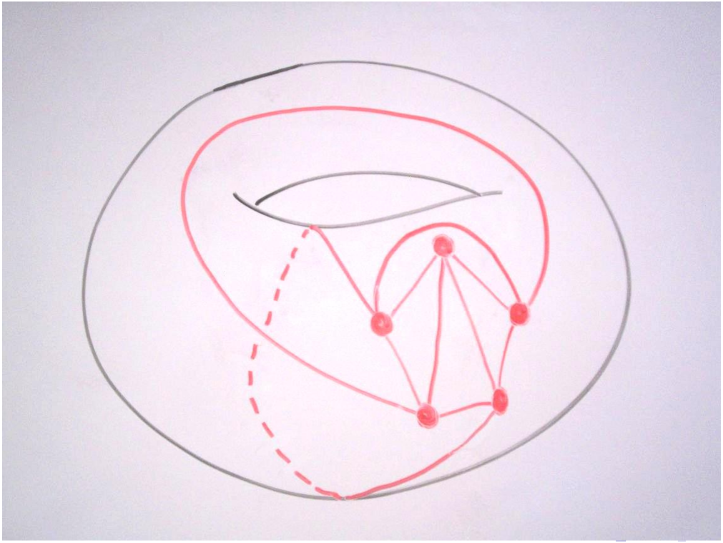

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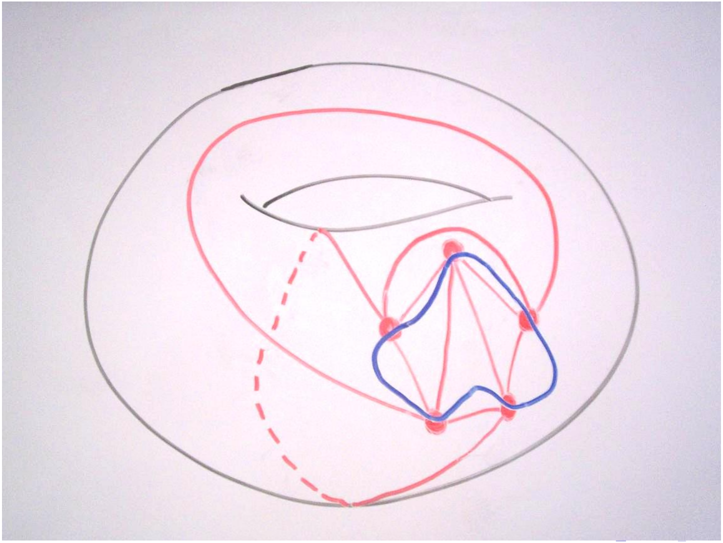

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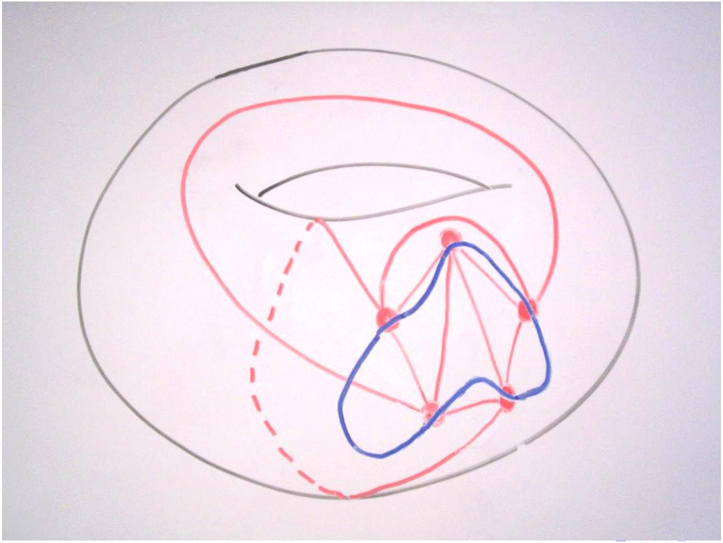

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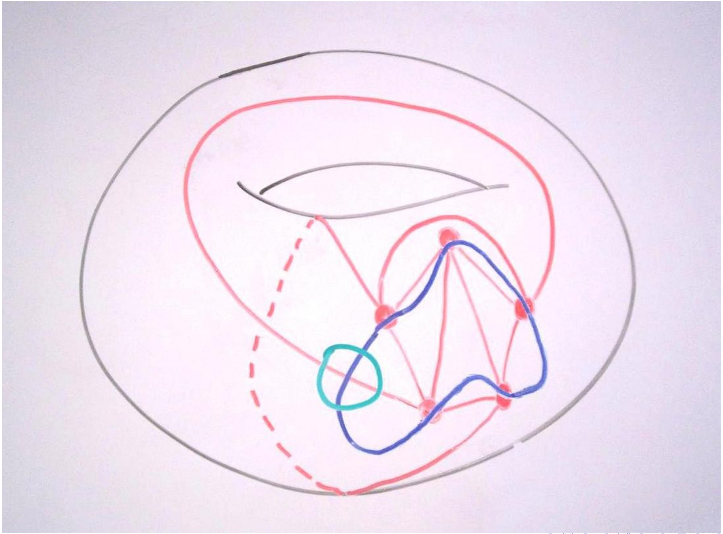

Sphere cut decompositions

Key idea for planar graphs [Dorn *et al.* *ESA'05*]:

- Sphere cut decomposition: Branch decomposition where the vertices in each **mid**(e) are situated around a noose. [Seymour and Thomas. *Combinatorica'94*]

- Recall that the size of the tables of a DP algorithm depends on how many ways a partial solution can intersect **mid**(e).
- In how many ways can we draw polygons inside a circle such that they touch the circle only on its k vertices and they do not intersect?

- Exactly the number of *non-crossing partitions* over k elements, which is given by the k-th Catalan number:

$$\text{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \approx 4^k.$$

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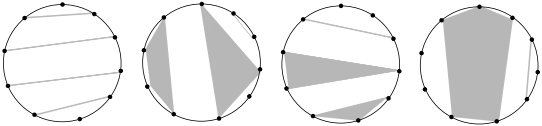

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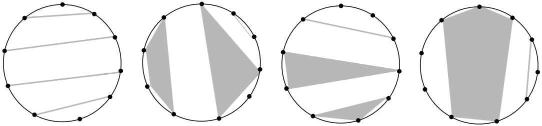

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“Old” idea for graphs on surfaces

Key idea for graphs on surfaces [Dorn *et al.* *SWAT’06*]:

- Perform a planarization of the input graph by splitting the potential solutions into a number of pieces depending on the surface.
- Then, apply the sphere cut decomposition technique to a more complicated version of the problem where the number of pairings is still bounded by some Catalan number.
- Drawbacks of this technique:
  - It depends on each **particular** problem.
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Key idea for graphs on surfaces [Dorn *et al.* *SWAT’06*]:

- Perform a planarization of the input graph by splitting the potential solutions into a number of pieces depending on the surface.
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From sphere to surface cut decompositions

Our approach is based on a new type of branch decomposition, called surface cut decomposition.

- Surface cut decompositions for graphs on surfaces **generalize** sphere cut decompositions for planar graphs.
[Seymour and Thomas. *Combinatorica'94*]

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Surface cut decompositions (simplified version)

Let G be a graph embedded in a surface Σ, with $\mathbf{eg}(\Sigma) = \mathbf{g}$.

A surface cut decomposition of G is a branch decomposition (T, μ) of G and a subset $A \subseteq V(G)$, with $|A| = \mathcal{O}(\mathbf{g})$, s.t. for all $e \in E(T)$

- either $|\mathbf{mid}(e) \setminus A| \leq 2$,
- or
  - $\star$ the vertices in $\mathbf{mid}(e) \setminus A$ are contained in a set $\mathcal{N}$ of $\mathcal{O}(\mathbf{g})$ nooses;
  - $\star$ these nooses intersect in $\mathcal{O}(\mathbf{g})$ vertices;
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Main results (I)

Surface cut decompositions can be efficiently computed:

Theorem (Rué, Thilikos, and S.)

Given a G on n vertices embedded in a surface of Euler genus $\mathbf{g}$*, with* $\mathbf{bw}(G) \leq k$*, one can construct in* $2^{3k+\mathcal{O}(\log k)} \cdot n^3$ *time a surface cut decomposition* (T, μ) *of G of width at most* $27k + \mathcal{O}(\mathbf{g})$.

Sketch of the construction of surface cut decompositions:

- Partition G into **polyhedral** pieces, plus a set of A vertices, with $|A| = \mathcal{O}(\mathbf{g})$.
- For each piece H, compute a branch decomposition, using Amir's algorithm.
- Transform this branch decomposition to a **carving** decomposition of the **medial** graph of H.
- Make the carving decomposition **bond**, using Seymour and Thomas' algorithm.
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- Construct a branch decomposition of G by **merging** the branch decompositions of all the pieces.

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After some study of bicolored trees and its asymptotics...

Theorem (Rué, Thilikos, S.)

Let Σ be a surface whose boundary has $\beta(\Sigma)$ connected components. Then the number of non-crossing partitions on Σ with k vertices is asymptotically bounded by

$$f(\Sigma) \cdot k^{-3/2\chi(\Sigma)+\beta(\Sigma)-1} \cdot 4^k,$$

where

- $f(\Sigma)$ *is a function depending only on* Σ.
- $\chi(\Sigma)$ *is the Euler characteristic of* Σ: $\chi(\Sigma) = 2 - \mathbf{eg}(\Sigma) - \beta(\Sigma)$.

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How to use this framework?

- We presented a framework for the design of DP algorithms on **surface-embedded** graphs running in time $2^{\mathcal{O}(k)} \cdot n$.

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  1. Let **P** be a **connected packing-encodable** problem on a surface-embedded graph G.
  2. As a **preprocessing** step, build a **surface cut decomposition** of G, using the 1st Theorem.
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Next section is...

Structure of minor-free graphs

- Idea: use the structure of minor-free graphs.
- Some (simplified) preliminaries:
  - h-clique-sum of two graphs G_1 and G_2:
    choose cliques $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ with $|V(K_1)| = |V(K_2)| = h$, identify them, and possibly remove some edges of that clique.
  - Apex in an embedded graph:
    add a vertex with any neighbors in the embedded graph.
  - Vortex of depth h in an embedded graph:
    paste a graph of pathwidth at most h in a face of the embedding.
- Structure theorem [Robertson and Seymour]:
  Fix a graph H. There exists a constant $h = f(|V(H)|)$ such that any H-minor-free graph G can be decomposed (in a tree-like way) into h-clique-sums from h-almost-embeddable graphs: obtained from graphs of genus at most h by adding at most h apices and at most h vortices of depth at most h.

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Extension to minor-free graphs

- Strategy: use an extension of ***surface cut decomposition*** in each almost-embeddable graph, and then merge them.
- The clique-sums and the apices are "easy" to deal with, but the vortices are more complicated...
- We can capture their combinatorial behavior with h-triangulations: partition in the disk in which no subset of $h+1$ blocks pairwise intersect. *(A non-crossing partition is a* 1*-triangulation.)*
- It is known that the $\#$ of h-triangulations on k elements satisfies

$$T_h(k) \quad \leq_{k\to\infty} \quad \frac{h!}{\pi^{h/2}} \cdot k^{-3h/2} \cdot 4^{hk}$$

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1. Motivation
2. Graphs on surfaces
   - Preliminaries
   - Main ideas of our approach
3. Extension to minor-free graphs
4. Conclusions
   - Further research
   - Some recent results

Next subsection is...

Further research

Can this framework be applied to **more complicated problems**?

Fundamental problem: H-MINOR CONTAINMENT

- ⋆ Minor containment for host graphs G on surfaces.
  [Adler, Dorn, Fomin, S., Thilikos. *SWAT'10*]

  With running time $2^{\mathcal{O}(k)} \cdot h^{2k} \cdot 2^{\mathcal{O}(h)} \cdot n$.

  ($h = |V(H)|$, $k = \mathbf{bw}(G)$, $n = |V(G)|$)

- ⋆ Single-exponential algorithm for planar host graphs.
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Rooted graph problems: DISJOINT PATHS

Next subsection is...

1. Motivation
2. Graphs on surfaces
   - Preliminaries
   - Main ideas of our approach
3. Extension to minor-free graphs
4. Conclusions
   - Further research
   - Some recent results

Lower bounds (I)

For an FPT problem, is it always possible to obtain algorithms with running time $c^k \cdot n^{\mathcal{O}(1)}$?

[Lokshtanov, Marx, Saurabh. *SODA'11*]:
Assuming that 3SAT cannot be solved in $2^{o(n)}$ time (ETH), then:

- Disjoint Paths cannot be solved in $2^{o(\mathbf{tw} \log \mathbf{tw})} \cdot n^{\mathcal{O}(1)}$ time.
- d-Distortion cannot be solved in $2^{o(d \log d)} \cdot n^{\mathcal{O}(1)}$ time.

Here, $\mathbf{tw} = \mathbf{tw}(G)$ and $n = |V(G)|$. These bounds are tight.

OPEN QUESTIONS:

- Parameterizing by treewidth: Hamiltonian Path, FVS, Connected Vertex Cover, Connected Dominating Set... Is $2^{\mathcal{O}(\mathbf{tw} \log \mathbf{tw})} \cdot n^{\mathcal{O}(1)}$ time optimal?
- Parameterizing by solution size: Directed FVS, Interval Completion... Is $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$ time optimal?

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- For any $q \geq 3$, q-Coloring cannot be solved in $(q-\varepsilon)^{\mathbf{tw}} \cdot n^{\mathcal{O}(1)}$.

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They present randomized algorithms for connected packing-encodable problems in general graphs with time $c^{\mathbf{tw}} \cdot n^{\mathcal{O}(1)}$.

- They introduce a dynamic programming technique called Cut&Count.
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